# ALGEBRAIC CURVES
# EXERCISE SHEET 10

Unless otherwise specified, $k$ is an algebraically closed field.

**Exercise** 1.

For $n, d \geq 1$, let $V(d,n)$ the $k$-vector space of forms of degree $d$ in $k[X_1, \dots, X_n]$.

(1) Compute $\dim_k(V(d,n))$ for $d \geq 1$ and $n = 1, 2, 3$. Can you find a formula for arbitrary $n$?

Set $n = 2$. Let $L_i$, $i \geq 1$ and $M_j$, $j \geq 1$ be two sequences of non-zero linear forms in $k[X,Y]$ such that $L_i \neq \lambda M_j$ for all $i, j \geq 1$, $\lambda \in k$. Consider $A_{ij} = L_1 \dots L_i M_1 \dots M_j$, $i, j \geq 0$ (if $i = 0$ or $j = 0$, the empty product is taken as 1).

(2) Show that $A_{ij}$, $i + j = d$, $i, j \geq 0$ form a basis of $V(d, 2)$. (Hint: think of dehomogenizing the $A_{ij}$ by setting $Y = 1$.)

**Exercise** 2.

Recall properties 1 to 9 of intersection numbers from the course (Thm. 4.5). Prove property 8 using only properties 1 to 7. (Hint: introduce a uniformizer $\varpi$ of $\mathcal{O}_P(F)$ and rewrite the factorization $G = u\varpi^n$, $u \in \mathcal{O}_P(F)^\times$ in terms of polynomials in $k[X,Y]$.)

**Exercise** 3.

Compute the intersection numbers at $P = (0,0)$ of various pairs of the following curves:

- $A = Y - X^2$
- $B = Y^2 - X^3 + X$
- $C = Y^2 - X^3$
- $D = Y^2 - X^3 - X^2$
- $E = (X^2 + Y^2)^2 + 3X^2Y - Y^3$
- $F = (X^2 + Y^2)^3 - 4X^2Y^2$

**Exercise** 4.

Consider the affine curves $F = Y - X^2$ and $L = aY + bX + c$, where $a, b, c \in k$ and $(a, b) \neq (0, 0)$.

(1) Compute the intersection points $P \subseteq F \cap L$ and their intersection numbers $I(P, F \cap L)$. Consider $s = \sum_P I(P, F \cap L)$. Give a necessary and sufficient condition for $s = 1$.

Let us identify $\mathbb{A}_k^2$ with the affine open subset $U_1 = \{x_1 \neq 0\} \subseteq \mathbb{P}_k^2$, where we use projective coordinates $x_1, x_2, x_3$. Consider $\overline{V}$ (resp. $\overline{L}$) the closure of $V(F) \subseteq U_1$ (resp. $V(L)$) in $\mathbb{P}_k^2$.

(2) Assume that $s = 1$. Show that $\overline{V}$ and $\overline{L}$ admit another intersection point outside $U_1$ and that the intersection number (computed in the affine plane $U_2$ or $U_3$) is 1.
(3) Same questions with $F = XY - 1$.

**Exercise** 5.

Let $F$ be an affine plane curve. Let $L$ be a line that is not a component of $F$. Suppose that $L = \{(a + tb, c + td),\ t \in k\}$. Define $G(T) = F(a + Tb, c + Td)$ and consider its factorization $G(T) = \epsilon \prod_i (T - \lambda_i)^{e_i}$ where the $\lambda_i$ are distinct.

(1) Show that there is a natural one-to-one correspondence between the $\lambda_i$ and the points $P_i \in L \cap F$.
(2) Show that, under this correspondence, $I(P_i, L \cap F) = e_i$. In particular, $\sum_i I(P_i, L \cap F) \leq \deg(F)$ (see for instance exercise 4).